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Derivation and implicit solution of the Harry Dym equation and its connections with the Korteweg–de Vries equation

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Abstract. The Harry Dym equation, which is related to the classical string problem, is derived in three different ways. An implicit cusp solitary-wave solution is constructed via a simple direct method. The existing connections between the Harry Dym and the Kortweg-de Vries equations are uniformised and simplified, and transformations between their respective solutions are carried out explicitly. Whenever possible, physical insights are provided.

1. Introduction

In trying to discover possible non-linear PDE that could be solved using our real exponential approach (Hereman *et al* 1985, 1986) we stumbled upon the Harry Dym (HD) equation $r_t = r^3 r_{3x}$ that has non-linearity and dispersion coupled together. This prototype of an evolution equation, which admits a cusp solitary wave solution (Wadati *et al* 1980) appears in many disguises, namely $r_t = (1-r)^3 r_{3x}$, $r_t = (r^{-1/2})_{3x}$ and $(r^2)_t = (r^{-1})_{3x}$. The first equation is occasionally referred to as the cusp-soliton equation (Kawamoto 1984a, b, 1985a). True, it failed to fit into our scheme but it certainly kindled our interest. According to some of the early references (Kruskal 1975, Magri 1978, Wadati *et al* 1979, 1980, Sabatier 1979a, b, Dijkhuis and Drohm 1979, Case 1982, Yi-Shen 1982, Calogero and Degasperis 1982) the origin of the HD lies in various private communications with Harry Dym. In search of truth, we contacted Dr Harry Dym who replied, 'In the spring of 1974,... Martin Kruskal delivered a few lectures on the isospectral theory.... Motivated by these lectures, I developed some analogues for the string equation. The HD equation, as Martin later termed it, was one of the outcomes.' Their collaboration resulted in developing a draft of 'a fairly complete theory' which is 'gathering dust' in Dr Kruskal's office.

From the *fact* files, the HD is a completely integrable non-linear evolution equation (Wadati *et al* 1979, 1980) which can be solved by the inverse scattering transform (IST). Dijkhuis and Drohm (1979) and Calogero and Degasperis (1982) discuss the HD equation as a special case of a new broad class of non-linear PDE tractable by IST. The HD has a bi-Hamiltonian structure (Magri 1978, Case 1982, Olver 1986); it possesses an infinite number of conservation laws and infinitely many symmetries (Magri 1978,

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Ibragimov 1985, Olver 1986); and reciprocal Bäcklund transformations (Rogers and Nucci 1986, Nucci 1988). However, the HD equation does not directly possess the Painlevé property (Weiss 1983, 1986, Steeb and Louw 1987, Fokas 1987, Hereman and Van den Bulck 1988) indicating that the Painlevé property is at most sufficient, but not necessary, for integrability. We should remark that for the HD case, and other types of equations which allow movable branch-point singularities, a natural extension of the Painlevé property is possible. Indeed, a process similar to the 'uniformisation' of algebraic curves allows us to transform non-Painlevé-type equations into P-type equations. Therefore, solutions of equations that lack the Painlevé property still may be implicitly representable in terms of meromorphic functions (Weiss 1986).

Yi-Shen (1982) and Sabatier (1979a, b, 1980) recalled the derivation of the HD equation. The first author started from the isospectral equation modelling the classical string problem with a varying elastic constant. Sabatier independently rediscovered the HD equation and its generalisations within the Lax formalism (Lax 1968). In order to make this paper a self-contained tutorial of the HD equation, we shall first review and simplify these calculations, and also include a heuristic derivation of the HD equation to provide some physical motivation.

Amongst the first researchers to study the connections between the HD and the KdV equations we mention Ibragimov (1981, 1985), Calogero and Degasperis (1982) and Weiss (1983, 1986). Later on Kawamoto (1985b), Rogers and Nucci (1986) and Steeb and Louw (1987) investigated the links to the MKdV equation. Underlying all these connections there exists a rigorous, though involved, Lie-Bäcklund algebra. Excellent references on this approach are in books by Ibragimov (1985) and Olver (1986). In keeping with a tutorial point of view we therefore recognise the need for uniformisation and simplification of these connections in order to provide a better insight. Also, as part of our simplifying efforts, we rederive the implicit solution to the HD equation in the simplest possible way, without the aid of heavy machinery like the IST. We obtain a particular exact solution which cannot, however, be expressed in closed form owing to the presence of a transcendental phase. Although the links between the equations have been extensively studied, limited attention has been paid to connections between the various solutions. In our opinion this is vital, since a customary technique in non-linear science is to generate solutions to a non-linear PDE from known solutions of another.

The organisation of this paper is as follows. In § 2 we first formally derive the HD equation from the string problem with a varying elastic constant and where the eigenvalue λ is constant WRT a parameter t. We next employ the Lax operator technique for this isospectral eigenvalue problem. Finally, we retrieve the HD equation by a heuristic method. This involves the derivation of the linear part of the PDE from the known dispersion relation, and suitable modification of the coefficients to account for non-linear effects.

As stated earlier, the HD equation is conventionally solved using IST. We present, in § 3, a novel direct integration method to construct the implicit cusp-type single solitary-wave solution of the HD equation. Guided by an *a priori* knowledge of the final result, we assume a form of the solution which is inherently implicit. Mathematically speaking, this is achieved by a change of variable that depends on the solution of the equation itself. As will be explained later, the introduction of this type of variable is commensurate with the physical basis for implicit solutions of other non-linear PDE, namely the non-linear kinematic equation (for other examples see, e.g., Wadati *et al* (1979, 1980) and Kawamoto (1984a, b, 1985a)).

Now, the HD equation has intriguing links with other non-linear evolution equations, namely the ubiquitous KdV equation, and a Liouville-type equation (Hereman and Banerjee 1988). In § 4, we therefore summarise some of the existing links between the HD equation and the KdV equation by transforming the one into the other and by providing the connections between the corresponding eigenvalue problems. Scattered work done by Ibragimov (1981, 1985), Weiss (1983, 1986) and Kawamoto (1985b) have contributed to the overall picture we draw in this section. The procedures we employ comprise the Bäcklund transformation method, a technique involving the Schwarzian derivative, and the Cole–Hopf and Miura transformations. Every procedure above needs to be augmented by either an explicit-to-implicit or an implicit-to-explicit transformation. It is thus not surprising that the single-solitary-wave solution to the HD equation is an implicit one. For a better understanding we compare, wherever possible, different steps in every conversion procedure. For instance, it is realised that the Schwarzian derivative, which occurs here in the context of Painlevé analysis, may be conceived of as a combination of the Miura and the Cole–Hopf transformations.

We next turn to transforming the known solution(s) of one equation to that of the other. This is accomplished in § 5. Of particular interest is the case where, by starting from the cusp-soliton solution of the HD equation, we are able to derive a new closed form, though singular, solution of the KdV equation.

2. Derivation of the Harry Dym equation

In this section we shall derive the HD equation by (*a*) starting from the classical string problem, (*b*) employing the Lax operator technique and (*c*) using a heuristic approach.

2.1. Derivation from classical string problem

Consider the ODE

$$\psi_{1,xx} = -\frac{\lambda}{r^2(x;\, t)}\,\psi_1 \tag{1}$$

which models the classical string problem where the string, for instance, has a varying elastic constant (Sabatier 1979a, b, 1980, Dijkhuis and Drohm 1979, Yi-Shen 1982, Calogero and Degasperis 1982). In (1), λ denotes the eigenvalue and $r(x;\, t)$ is a bounded positive function of x. Furthermore, we assume that $r(x;\, t) \to 1$ as $|x| \to \infty$. For $r(x;\, t) = 1$, equation (1) reduces to the standard Schrödinger equation. If $r(x;\, t) \neq 1$, equation (1) can be transformed back to that standard equation by a suitable change of variables (see § 4). Both r and ψ_1 depend on the parameter t. In order to find an integral representation for the solution of (1) that also incorporates these conditions, we first reduce (1) to an equivalent system of first-order equations. Defining $\psi_2 = \psi_{1,x}$, equation (1) may then be rewritten as

$$\Psi_x = M\Psi \tag{2a}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 \\ -\lambda/r^2 & 0 \end{pmatrix}. \tag{2b}$$

We remark here that the above decomposition is not unique in the sense that the *state variables* ψ_1 and ψ_2 could be chosen differently.

It was noted by Schlesinger (1912) that solvable non-linear PDE arise as the integrability conditions when a linear ODE as in (2) is deformed to, for instance,

$$\Psi_t = N\Psi \qquad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

in such a way as to preserve some *characteristic* (for instance, the spectrum λ) of the equation. Thus, setting $\lambda_t = 0$ (the isospectral case) the integrability conditions

$$\Psi_{xt} = \Psi_{tx} \tag{4}$$

yield the structure equation

$$M_t - N_x + [M, N] = 0 \tag{5}$$

where we define the commutator $[M, N] = MN - NM$. Incorporating the explicit forms for M and N (from (2b) and (3)) leads to four coupled equations, one of which describes the evolution of $r(x, t)$:

$$C + (\lambda/r^2)B = A_x \tag{6a}$$

$$D - A = B_x \tag{6b}$$

$$\lambda\left(2\frac{r_t}{r^3} - \frac{A}{r^2} + \frac{D}{r^2}\right) = C_x \tag{6c}$$

$$-(\lambda/r^2)B - C = D_x. \tag{6d}$$

From (6a) and (6d) it follows that

$$D = -A \tag{7}$$

where we have neglected a constant of integration. Using (6a, b) and (7) in (6c), the evolution of r may be rewritten as

$$\frac{r_t}{r^3} = -\frac{1}{4\lambda} B_{3x} + \frac{r_x}{r^3} B - \frac{B_x}{r^2}. \tag{8}$$

The choice $B = -4\lambda r$ (Yi-Shen 1982) leads straightforwardly to the HD equation:

$$r_t = r^3 r_{3x}. \tag{9}$$

2.2. *Derivation employing Lax operator technique*

Following Lax (1968), we can derive an eigenvalue problem of the type $L\tilde{\psi} = \lambda\tilde{\psi}$ with $\lambda_t = 0$ for a non-linear evolution equation expressible in the form

$$L_t = [B, L] \tag{10}$$

where L and B are linear spatial operators (respectively symmetric, hence self-adjoint, and antisymmetric) which may depend on $r(x, t)$ and its spatial derivatives. Now the ODE (1) may be modified to

$$\left(\frac{1}{r^{m-2}} \frac{\partial^2}{\partial x^2} r^m\right)\tilde{\psi} + \lambda\tilde{\psi} = 0 \tag{11}$$

where $\tilde{\psi}$ is a new wavefunction defined as $\tilde{\psi} = \psi_1/r^m$, and where m is an integer. From the constraint of self-adjointness, it readily follows that $m = 1$, yielding

$$L = r\frac{\partial^2}{\partial x^2} r \tag{12}$$

so that (10) becomes

$$\frac{r_t}{r}L + L\frac{r_t}{r} = [B, L]. \tag{13}$$

To construct the antisymmetric operator B we set, following Sabatier (1979b),

$$B = AL + LA \tag{14}$$

where A is antisymmetric; thus $[B, L] = [A, L]L + L[A, L]$. Comparing with (13), it follows that

$$[A, L] = r_t/r \tag{15}$$

provided the LHS of the above equation reduces to a multiplicative operator and (15) yields the required non-linear PDE. Noting that

$$L = \left(r\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}r\right) \overset{\text{def}}{=} ST \tag{16}$$

we set $A = S + T$, so that

$$\begin{aligned}[A, L] &= S[S, T] - [S, T]T \\ &= [S, [S, T]] + [S, T](S - T) \\ &= r(rr_{2x})_x - rr_x r_{2x} \\ &= r^2 r_{3x}\end{aligned} \tag{17}$$

where we have used the observations $T - S = r_x$ and $[S, T] = rr_{2x}$. From (15) and (17), the HD equation (9) follows.

2.3. *Derivation using heuristic approach*

The heuristic approach to constructing non-linear evolution and wave equations involves deriving the linear PDE from a (known) dispersion relation and, thereafter, incorporating non-linear corrections to the (linear) phase velocity c_0 (Korpel and Banerjee 1984). Consider the dispersion relation

$$\omega = c_0 k - \gamma k^3 \tag{18}$$

where ω denotes the angular frequency and k is the propagation constant. The linear PDE is obtained through replacing ω and k by their respective operators $\omega \to -\mathrm{i}\partial/\partial\tilde{t}$, $k \to \mathrm{i}\partial/\partial\tilde{x}$ in (18):

$$\tilde{r}_{\tilde{t}} + c_0\tilde{r}_{\tilde{x}} + \gamma\tilde{r}_{3\tilde{x}} = 0. \tag{19}$$

The non-linear extension is now done by assuming that γ, instead of c_0, is a non-linear function of $\tilde{r}$:

$$\gamma = \gamma_3\tilde{r}^3 \tag{20}$$

where γ_3 is constant. Incorporating this in (19) together with the scalings

$$x = \tilde{x} - c_0\tilde{t} \qquad t = \tilde{t} \qquad r(x, t) = -(\gamma_3)^{1/3}\tilde{r}(\tilde{x}, \tilde{t}) \tag{21}$$

we readily retrieve the HD equation (9).

3. Derivation of the implicit single solitary-wave solution

In this section we employ direct integration to construct the (implicit) one solitary-wave solution of the HD equation previously found by Wadati *et al* (1980) using the IST. We are guided by the fact that the solitary-wave solution has a singularity in its derivative at its (bounded) peak value owing to the presence of a transcendental phase

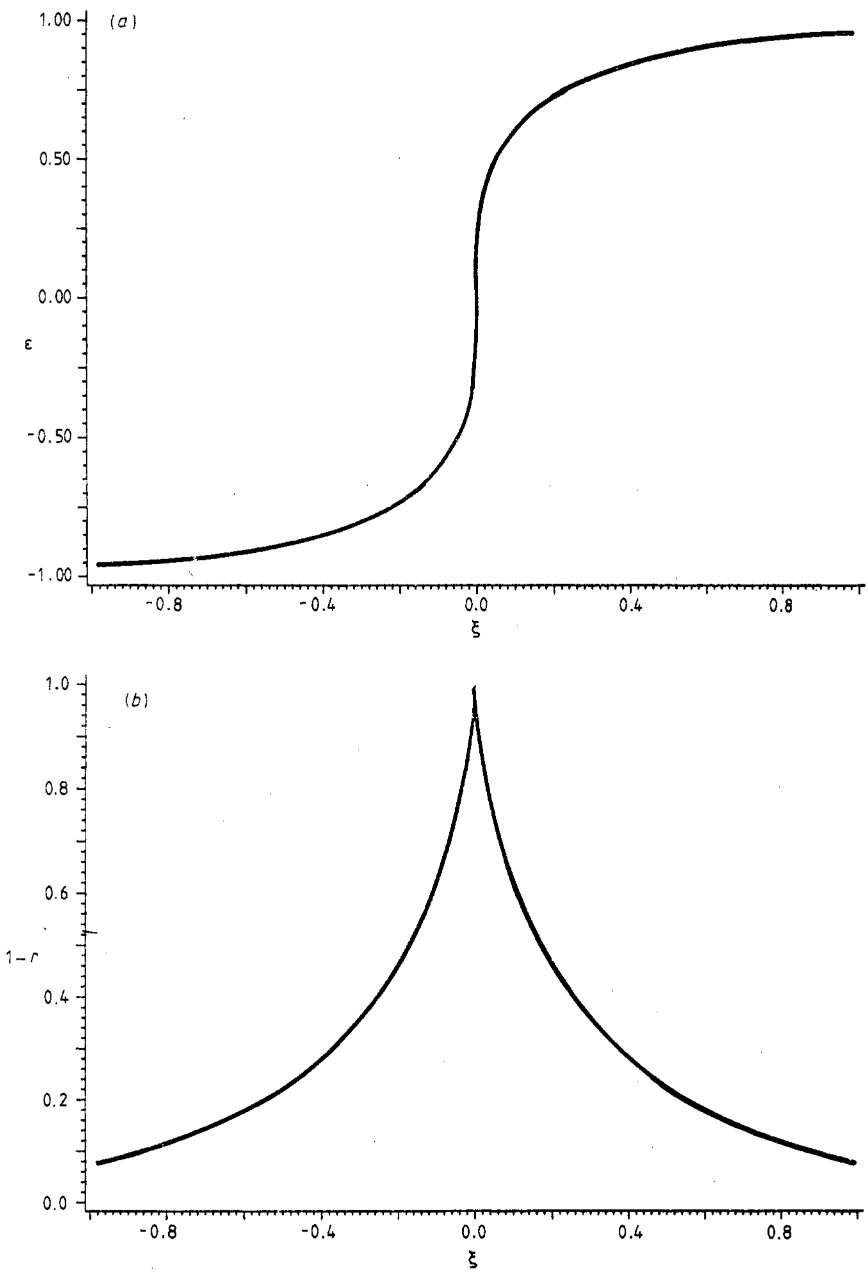

Figure 1. Plots for (a) ε and (b) $1-r$ against ξ where $\xi = x - x_0 - 4t$.

$\varepsilon(x, t)$ (see figure 1). We therefore seek a solution to (9) of the form

$$r(x, t) = F(f) \tag{22a}$$

where

$$f(x, t) = K(x - x_0 + Vt) + K\varepsilon(x, t) \tag{22b}$$

with

$$K\varepsilon(x, t) = G(f). \tag{22c}$$

In the above scheme, x_0 is a constant, K is the wavenumber and V is the K-dependent anticipated velocity of the solitary wave.

At this point, we wish to speculate on the physical basis of implicit solutions of non-linear PDE in general. Readers are reminded here of the real exponential approach to solving non-linear evolution and wave equations (Korpel 1978, Lambert and Musette 1984, Hereman *et al* 1985, 1986) where the final solution is assumed to be built up from the non-linear mixings of real exponential solutions to the *linear dispersive* part of the PDE. Alternatively, we may think of constructing a particular solution from the solution to the *non-linear non-dispersive* part of the PDE. This is a valid conjecture, since the non-linear non-dispersive part of, say, the KdV equation in $u(X, T)$, i.e. the non-linear kinematic equation (Whitham 1974)

$$u_T + \alpha u u_X = 0 \tag{23}$$

possesses shock-wave solutions that are intrinsically implicit:

$$u(X, T) = g(X - \alpha u(X, T)T). \tag{24}$$

In a more general sense we may therefore think of solutions to the entire non-linear dispersive PDE to be of the form (Banerjee and Hereman 1988a)

$$u(X, T) = F(f) \tag{25a}$$

with

$$f(X, T) = H_1(f)X - H_2(f)T + H_3(f). \tag{25b}$$

Examining (22), we note that it fits the scheme described above.

Returning to the construction of the implicit solution to (9), we first note from (22) that

$$\frac{\partial}{\partial t} = \left(\frac{KV}{1 - G_f}\right)\frac{\mathrm{d}}{\mathrm{d}f} \qquad \frac{\partial}{\partial x} = \left(\frac{K}{1 - G_f}\right)\frac{\mathrm{d}}{\mathrm{d}f}. \tag{26}$$

Equation (9) then transforms to

$$\frac{V}{K^2}\frac{F_f}{F^3} - \frac{\mathrm{d}}{\mathrm{d}f}\left[\left(\frac{1}{1 - G_f}\right)\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{F_f}{1 - G_f}\right)\right] = 0. \tag{27}$$

A first integration, followed by multiplication by F_f and a subsequent integration, yields

$$F_f^2 = c_1(1 - G_f)^2 F + c_2(1 - G_f)^2 + \frac{V}{K^2}\frac{(1 - G_f)^2}{F} \tag{28}$$

where c_1, c_2 are integration constants.

To solve for F, we must first eliminate G from (28) by assuming a relation between G_f and F. The obvious choice,

$$1 - G_f = F \tag{29}$$

reduces (28) to an ODE which can be readily solved either by employing the real exponential approach (Hereman *et al* 1985, 1986) or by direct integration. Using the latter, we get

$$f=\int\left(\frac{V}{K^2}F+c_2F^2+c_1F^3\right)^{-1/2}\mathrm{d}F \tag{30}$$

where the integration constant can be absorbed in x_0. For the choice

$$c_1=V/K^2 \qquad c_2=-2V/K^2 \tag{31}$$

equation (30) may be readily evaluated as

$$F(f)=\tanh^2\left(\frac{\sqrt{V}}{2K}f\right). \tag{32a}$$

From (29) we then obtain

$$G(f)=\frac{2K}{\sqrt{V}}\tanh\left(\frac{\sqrt{V}}{2K}f\right). \tag{32b}$$

For later use, we write down the implicit solution to the HD equation in the original variables x and t:

$$r(x,t)=\tanh^2\left(\frac{\sqrt{V}}{2}(x-x_0+Vt+\varepsilon(x,t))\right) \tag{33a}$$

with

$$\varepsilon(x,t)=\frac{2}{\sqrt{V}}\tanh\left(\frac{\sqrt{V}}{2}(x-x_0+Vt+\varepsilon(x,t))\right). \tag{33b}$$

This type of cusp-soliton solution (which is plotted in figure 1 for $V=4$) has also been obtained for coupled systems of evolution equations. Kawamoto (1984a, b, 1985a) discusses two examples: an Ito-type system and a normalised Boussinesq equation.

4. Connections between the HD and KdV equations

Having derived the cusp-solitary wave solution to the HD equation, we would like to investigate how this solution maps to particular (perhaps new) solutions of the KdV equation. Toward this goal we first reconstruct the transformations between the HD and KdV equations according to the following scheme:

(a) by deriving the HD from the KdV equation using (i) the Bäcklund transformation obtained by Weiss (1983, 1986) and (ii) the Schwarzian transformation proposed by Ibragimov (1981, 1985);

(b) by rederiving the KdV equation from the HD equation using the Cole–Hopf and Miura transformations as suggested by Kawamoto (1985b);

(c) by transforming the spectral problem associated with the HD equation to the one for the KdV equation (Calogero and Degasperis 1982).

We remark that, in addition to every transformation above, we need an *explicit* (*implicit*) to *implicit* (*explicit*) transformation in going from the KdV (HD) equation to the HD (KdV) equation. This explains why the solitary-wave solution of the HD equation is implicit in nature. Detailed calculations on the conversion of the cusp-soliton solution of the HD equation to particular solutions of the KdV equation (and vice versa) will be presented in § 5.

4.1. *Transformation from the* KdV *to* HD

4.1.1. Bäcklund transformation method. This conversion process from the KdV equation to the HD equation can be summarised in the following three steps.

(i) We first use the auto-Bäcklund transformation

$$u(X, T)=(12/\alpha)[\ln \phi(X, T)]_{2X}+u_2(X, T) \tag{34}$$

derived by Weiss (1983, 1986) in the context of the Painlevé analysis of the KdV equation

$$u_T+\alpha u u_X+u_{3X}=0. \tag{35}$$

The function $\phi(X, T)$ plays a crucial role as a *singular* or *pole* manifold (Newell *et al* 1987) in the Painlevé formalism and in Hirota's bilinear method as a new dependent variable (Matsuno 1984, Gibbon *et al* 1985). We remark that $u_2(X, T)$, like $u(X, T)$, must also be a solution to (35). Hence, from subtracting the respective equations, we obtain

$$(u-u_2)_T+(u-u_2)_{3X}+\tfrac{1}{2}\alpha[(u-u_2)^2+2u_2(u-u_2)]_X=0. \tag{36}$$

We now substitute for $u-u_2$ from (34) into (36) and integrate once WRT X. After a little algebra, and equating the coefficients of ϕ^{-1} and ϕ^{-2}, we obtain the two non-trivial relations:

$$\phi_{XT}+\alpha\phi_{2X}u_2+\phi_{4X}=0 \tag{37a}$$

$$\phi_X\phi_T+\alpha\phi_X^2u_2+4\phi_X\phi_{3X}-3\phi_{2X}^2=0. \tag{37b}$$

The coefficients ϕ^{-3} and ϕ^{-4} identically vanish. Upon elimination of u_2 from (37), and one integration WRT X, we get

$$\phi_T/\phi_X+\{\phi; X\}=\mu \tag{38}$$

where the Schwarzian derivative (Hille 1976) is defined by

$$\{\phi; X\}=\phi_{3X}/\phi_X-\tfrac{3}{2}(\phi_{2X}/\phi_X)^2. \tag{39}$$

In (38) μ may depend on T but is constant in X. We recall the property of *Galilean invariance* of (38): if $\phi(X, T; \mu)$ denotes the solution to (38), then

$$\phi(X, T; 0)=\phi(X-\mu T, T; \mu). \tag{40}$$

(ii) This step involves the explicit-to-implicit transformation by interchanging the dependent variable, ϕ, and the independent variable X. This is effected by first recalling a property of the Schwarzian derivative (Hille 1976), namely

$$\{\phi; X\}=-\phi_X^2\{X; \phi\}. \tag{41}$$

Together with the observations

$$\phi_X=1/X_\phi \qquad T \text{ constant} \tag{42}$$

and

$$\phi_T=\phi_X X_T=X_T/X_\phi \tag{43}$$

(38) may be recast into the form

$$X_T=\mu+(1/X_\phi^2)\{X; \phi\} \tag{44}$$

with $X=X(\phi, T)$.

(iii) Finally, we define a new dependent variable

$$w = 1/X_\phi \tag{45}$$

and study the evolution of $w(\phi, T)$. We note that

$$w_T = -X_{\phi T}/X_\phi^2 = -w^2 X_{\phi T} \tag{46}$$

which can be re-expressed, using (44), as

$$\begin{aligned} w_T &= -\mu_\phi w^2 + w^2(w w_{2\phi} - \tfrac{1}{2} w_\phi^2)_\phi \\ &= w^3 w_{3\phi} - \mu_\phi w^2. \end{aligned} \tag{47}$$

Consider the special case where $\mu_\phi = 0$. Upon defining new variables

$$x = \phi \qquad t = T \qquad r(x, t) = w(\phi, T) \tag{48}$$

we readily obtain the HD equation (9).

4.1.2. The Schwarzian transformation method. The next conversion method from the KdV equation to the HD equation is partly motivated by the inverse transformation due to Kawamoto (1985b). In essence, Kawamoto's procedure involves an implicit-to-explicit transformation to reduce the HD equation as in (9) to an intermediate equation similar to (38), and then using the Cole–Hopf transformation on ϕ_X to yield the MKdV equation. The Miura transformation finally links the MKdV equation to the KdV equation.

To go from the KdV equation to the HD equation we will apply the above transformations in reverse order to yield the intermediate equation (38) with $\mu = 0$. In that sense, this method differs from the Bäcklund transformation method in the first step. We first use the Miura transformation to $v(X, T)$:

$$u = v^2 + \mathrm{i}(6/\alpha)^{1/2} v_X \tag{49a}$$

followed by the Cole–Hopf transformation on the potential $\phi_X(X, T)$:

$$V = -\mathrm{i}(3/2\alpha)^{1/2} \phi_{2X}/\phi_X. \tag{49b}$$

After some tedious algebra, which has been performed using MACSYMA, we can identify the relation

$$L_1(\phi_T + \{\phi; X\}\phi_X) = 0 \tag{50}$$

where the operator L_1 is given by

$$L_1 = \frac{3}{\alpha}\left[\frac{1}{\phi_X}\frac{\partial^3}{\partial X^3} - 3\frac{\phi_{2X}}{\phi_X^2}\frac{\partial^2}{\partial X^2} - \left(\frac{\phi_{3X}}{\phi_X^2} - 3\frac{\phi_{2X}^2}{\phi_X^3}\right)\frac{\partial}{\partial X}\right] \tag{51}$$

from which we recover the intermediate equation (38) with $\mu = 0$. With MACSYMA we have obtained yet another but equivalent representation of (50):

$$L_2[(\phi_T/\phi_X) + \{\phi; X\}] = 0 \tag{52}$$

with

$$L_2 = \frac{3}{\alpha}\left(\frac{\partial^3}{\partial X^3} + 2\{\phi; X\}\frac{\partial}{\partial X} + \{\phi; X\}_X\right). \tag{53}$$

The latter operator is similar to the antisymmetric Lax operator for the KdV equation, wherein we have formally replaced $u(X, T)$ by a multiple of $\{\phi; X\}$.

Note that the successive transformations defined in (49) may indeed be combined to give

$$u = (3/\alpha)\{\phi; X\} \tag{54}$$

which is the Schwarzian transformation proposed by Ibragimov (1981, 1985). The argument presented in Ibragimov's book may be summarised as follows. Suppose there exists a transformation from (38) to the form

$$\Psi_T + \Psi_{3X} + \Phi(\Psi, \Psi_X) = 0 \tag{55}$$

having a non-trivial Lie-Bäcklund algebra. Then (38) is equivalent to either a linear equation with constant coefficients or the KdV equation. In our case, (49*b*) provides the relevant transformation to the MKdV equation where $\Psi = v$; $\Phi = \beta v^2 v_X$, which is equivalent to the KdV through the Miura transformation (49*a*).

The subsequent steps for the transformation are identical to steps (ii) and (iii) in the Bäcklund transformation method.

4.2. Transformation from the HD *equation to* KdV *equation*

Along the lines of Kawamoto (1985b), we convert the HD equation to the KdV equation through the following three steps.

(i) We first employ an implicit-to-explicit transformation by defining

$$X = \int_{-\infty}^{x} \frac{\mathrm{d}s}{r(s, t)} \qquad T = -t \tag{56}$$

with $R(X, T) = r(x(X, T), t(X, T))$ representing the new transformed dependent variable. We observe that

$$\begin{aligned}\frac{\partial}{\partial t} &= -\frac{\partial}{\partial T} + (-rr_{2x} + \tfrac{1}{2}r_x^2)\frac{\partial}{\partial X}\\ &= -\frac{\partial}{\partial T} - \left(\frac{R_{2X}R - \frac{3}{2}R_X^2}{R^2}\right)\frac{\partial}{\partial X}\end{aligned} \tag{57a}$$

where use has been made of the HD equation (9) to replace all time derivatives of r in terms of spatial derivatives, and that

$$\frac{\partial}{\partial x} = \frac{1}{R}\frac{\partial}{\partial X}. \tag{57b}$$

After a little algebra, (9) can be expressed as

$$R_T + \frac{R_{3X}R^2 - 3R_{2X}R_X R + \frac{3}{2}R_X^3}{R^2} = 0. \tag{58}$$

In the above derivation we have used the fact that $r(x, t)$ tends to one and its spatial derivatives tend to zero as $|x| \to \infty$.

Before proceeding any further, we make the following remarks.

Remark 1. The substitutions in (56) are equivalent to

$$X = \int_{x}^{\infty} \frac{\mathrm{d}s}{r(s, t)} \qquad T = t. \tag{59}$$

Remark 2. Comparing (58) with (38), it may be readily recognised that

$$R = \phi_X. \tag{60}$$

(ii) We now write

$$v = R_X / R \tag{61}$$

relating v to ϕ by a Cole-Hopf transformation, and find $v_T(X, T)$ with the use of (58). It is easy to check that v_T can be expressed entirely in terms of v and its spatial derivatives through the MKdV equation

$$v_T - \tfrac{3}{2}v^2 v_X + v_{3X} = 0. \tag{62}$$

(iii) The Miura transformation

$$u = -(3/2\alpha)(v^2 + 2v_X) \tag{63}$$

now relates (62) to the KdV equation (35). This may be readily verified by computing u_T from (63), using (62) to replace the time derivative(s) of v in terms of v and its spatial derivatives, and finally re-expressing in terms of u.

4.3. *Connection between the respective eigenvalue problems*

We will complete the connection between the HD and the KdV equations by pointing out the link between their respective eigenvalue problems. To this end, we first employ the implicit-to-explicit transformation described by (56), which is similar to the transformation employed by Calogero and Degasperis (1982). The eigenvalue problem (1) for the HD equation is then equivalent to

$$(\theta_{2X} R - \theta_X R_X)/R = -\lambda\theta \tag{64}$$

where $\theta(X; T) = \psi_1(x(X, T); t(T))$. Now defining

$$\Omega(X; T) = R^{-1/2}\theta(X; T) \tag{65}$$

and eliminating θ from (64), we obtain

$$\Omega_{2X} + [\tfrac{1}{2}(R_{2X}/R) - \tfrac{3}{4}(R_X^2/R^2) + \lambda]\Omega = 0 \tag{66a}$$

or, using (60),

$$\Omega_{2X} + (\tfrac{1}{2}\{\phi; X\} + \lambda)\Omega = 0. \tag{66b}$$

Note that, if we adopt the Cole-Hopf transformation (61), then (66*a*) may be also written as

$$\Omega_{2X} + [\lambda - \tfrac{1}{4}(v^2 - 2v_X)]\Omega = 0 \tag{67}$$

and thence as

$$\Omega_{2X} + (\lambda - \tilde{u})\Omega = 0 \tag{68}$$

provided $\tilde{u}(X, T)$ is a solution to the KdV equation

$$\tilde{u}_T - 6\tilde{u}\tilde{u}_X + \tilde{u}_{3X} = 0. \tag{69}$$

Thus, we have transformed the eigenvalue problem for the HD equation to that for the KdV equation, where the corresponding eigenfunctions $\psi_1(=\theta)$ and Ω are related through (65). The explicit form for $R(X, T)$ will be discussed in the following section.

5. Connections between the solutions of the HD and KdV equations

Using the knowledge of the connections between the HD and KdV equations discussed in the previous section, we shall proceed to transform the known solution(s) of each equation to the solution(s) of the other. The transformation of the implicit solution of the HD equation to explicit solution(s) of the KdV equation is more interesting and will be taken up first.

We shall first apply the implicit-to-explicit transformation prescribed by (56) to the cusp-solitary wave solution of the HD equation as in (33). From the latter equation

$$1/r = 1 + \varepsilon_x. \tag{70}$$

Hence, after modifying (56) to an indefinite integral,

$$X = x + \varepsilon(x, t) + X_0 \tag{71}$$

where X_0 is an integration constant.

From (33*a*) and the definition of R as in (56), we obtain

$$R(X, T) = \tanh^2[(\sqrt{V}/2)(X - X_0 - VT)] \tag{72}$$

where the constant x_0 has been absorbed into X_0.

For later use, we will write down the expression for ϕ from (60). Using (72), and one integration WRT X, we get

$$\phi = X - X_1(T) - (2/\sqrt{V}) \tanh[\tfrac{1}{2}\sqrt{V}(X - X_0 - VT)] \tag{73}$$

where $X_1(T)$ is an integration constant.

From (61) and (72), we arrive at the singular solution to the MKdV equation (62):

$$v(X, T) = 2\sqrt{V} \operatorname{cosech}[\sqrt{V}(X - X_0 - VT)]. \tag{74}$$

The solution to the KdV equation (35) may now be calculated using the Miura transformation (63). Straightforward algebra yields the familiar soliton solution

$$u(X, T) = (3V/\alpha) \operatorname{sech}^2[\tfrac{1}{2}\sqrt{V}(X - X_0 - VT)]. \tag{75}$$

An alternative solution of the KdV equation may be derived by starting from (73) and employing the auto-Bäcklund representation of u in terms of ϕ, as in (34). Indeed, since ϕ should also satisfy (38), it is readily checked, upon substituting (73), that $X_1(T) = \text{constant} = X_1$ and $\mu = 0$. Now, u_2 can be computed using either of the relations (37*a*) or (37*b*). This gives again the one-solitary-wave solution for u_2, as in (75). Returning to (34), a new, but singular, solution to the KdV equation is

$$u(X, T) = -\frac{3V}{\alpha}\left(\frac{[\frac{1}{4}V(X - X_1)^2 + 1]\tanh^2[\frac{1}{2}\sqrt{V}(X - X_0 - VT)] - \frac{1}{4}V(X - X_1)^2}{\{\tanh[\frac{1}{2}\sqrt{V}(X - X_0 - VT)] - \frac{1}{2}\sqrt{V}(X - X_1)\}^2}\right). \tag{76}$$

The evolution in time (for $T = 0$, $\frac{1}{16}$ and $T \to \infty$) of this solution has been plotted in figure 2; for simplicity we took $X_0 = X_1 = 0$ and $V = 4$. We remark that the inverse process of transforming the familiar sech^2 solution of the KdV equation to that of the HD equation leads to, for instance, a rational solution $r(x, t) = \frac{1}{2}\sqrt{V}(x - 1)$. Other possible solutions to the HD and KdV equations are presently under investigation (Banerjee and Hereman 1988b).

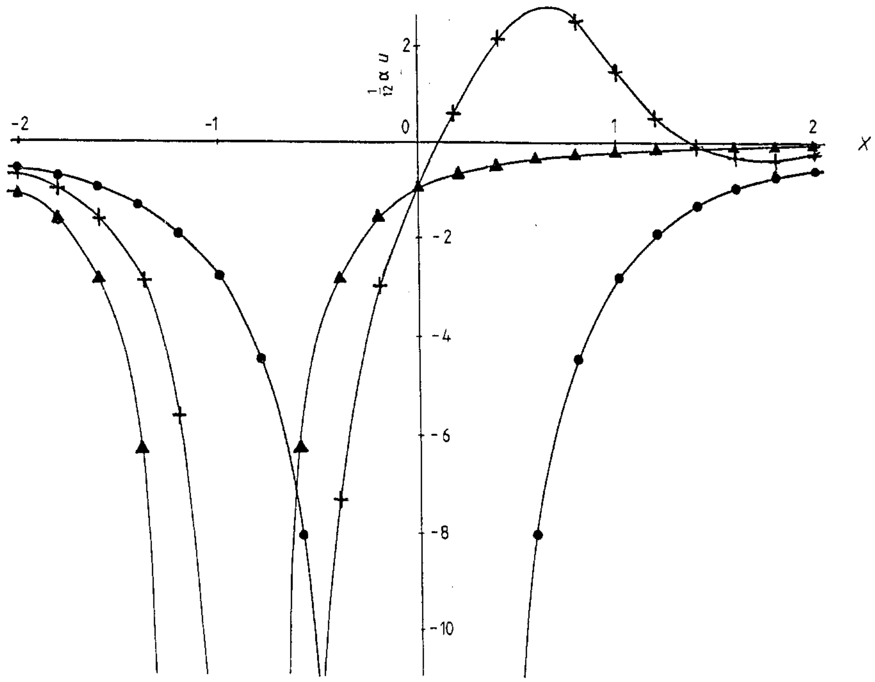

Figure 2. Evolution in time of $u(X, T)$ as in (76) for: $T=0$ (●); $T=\frac{1}{16}$ (+); $T\to\infty$ (▲).

6. Conclusions

In retrospect, we have provided some straightforward derivations of the HD equation. A direct integration method was used to derive a simple particular implicit solution. We have consolidated the various links between the HD, KdV and MKdV equations, and used these to provide connections between their solutions.

Although at this stage, the HD equation is more of theoretical significance than of applicative relevance, we hope that this paper contributes not only to provide a better insight into cusp solitary waves, but also serves as a review on the HD equation. The straightforward mathematical techniques employed throughout this paper should prove helpful in deriving implicit solutions to other non-linear PDE and in (as a spin-off) discovering new solutions to the KdV equation. Work on this is currently in progress.

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